Differential Topology — June 2024

Back-paper Exam — 100 Points

Notes.

(a) $\mathbb{R}$ = real numbers

(b) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

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1. [9 + 9 = 18 Points] In each case below, prove or disprove that X is a k-manifold with boundary for some $k \geq 0$. If so, identify k and the boundary of X. Briefly justify your answer.
   (i) X = the closed ball $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$
   (ii) X = the set $\{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$.

2. [5 + 5 + 8 + 8 = 26 Points] Let $f\colon X \to Y$ be a smooth map of boundaryless manifolds.
   (i) Define what it means for f to be an immersion.
   (ii) Define what it means for f to be a submersion.
   (iii) Give an example where f is an immersion and a submersion but not a diffeomorphism.
   (iv) Suppose f is a diffeomorphism. Prove that for any point $y \in Y$, $f^{-1}(y)$ is a submanifold. What are the possible dimensions of $f^{-1}(y)$?

3. [16 Points] Identify the regular and critical values of the function $f(x, y, z) = x^2 + y^2 - z^2$. Identify the tangent space at $p = (1, 1, 1)$ of the manifold $f^{-1}(f(p))$ as a linear subspace of $\mathbb{R}^3$.

4. [6 + 6 + 6 + 6 = 24 Points] Let X, Y, Z denote boundaryless manifolds.
   (i) Define what it means for a smooth map $f\colon X \to Y$ to be transversal to a submanifold $Z \subset Y$.
   (ii) Define what it means for two submanifolds X, Z in Y to intersect transversally.
   (iii) Give an example of two curves in $\mathbb{R}^2$ that don't intersect transversally.
   (iv) Explain why any two curves in $\mathbb{R}^3$ intersect transversally iff their intersection is empty.

5. [8 Points] Prove that for any compact manifold X, any smooth map $f\colon X \to \mathbb{R}$ has at least two critical points unless X is a point.

6. [8 Points] State without proof, the classification (upto diffeomorphism) of all connected 1-manifolds without boundary.